Mimetic Finite Difference Discretization of Diffusion-type Problems on Unstructured Polyhedral Meshes

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A numerical simulation is cheaper than a real-life experiment. It implies cheaper production cost in many industries (automobile building, oil production, etc). But the numerical simulations should be reliable, so that the predictions and insights gained from them were trustworthy. The determining factor for the reliability are accurate and efficient discretization methods.

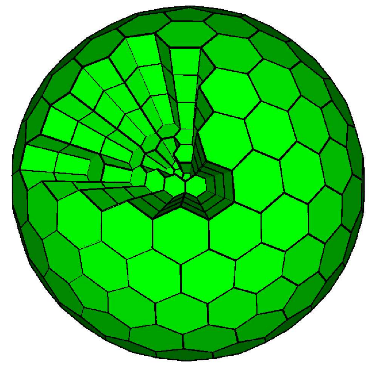
The main goal of this summer project is the investigation and the program realization of mimetic finite difference algorithms for a diffusion-type equation on unstructured polyhedral meshes. This problem can be formulated as the first order system:

$$\mathbf{F} = -K\,\mathrm{grad}\,p$$
$$\mathrm{div}\,\mathbf{F} + cp = Q$$

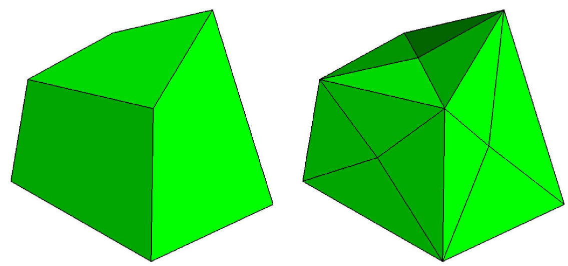
where K is a symmetric uniformly positive definite tensor, Q is a given source function, and p and $\mathbf{F}$ are unknown scalar and vector functions, respectively.

In many applications meshes with general type elements are more preferable than standard tetrahedral or cubic meshes. The polyhedral meshes appear in the basin modeling where mesh cells have to approximate sophisticated geological structures (faults, pinch-outs). For instance, a lot of applications in the computational fluid dynamics operate with Voronoi-type meshes.

Another example comes from applications using the adaptive mesh refinement algorithms or non-matching meshes. Specific situation occurs when we deal with polyhedrons which faces are extremely curve. In that case a curve face is approximated by a piecewise linear surface.



Example of a Voronoi-type mesh. The mesh consists of penta and hexa prisms and pyramids



Distorted cube with 6 curvilinear faces is transformed to polyhedron with 24 *planar faces*

Mimetic finite difference discretization uses discrete operators ($\mathcal{GRAD}$ and $\mathcal{DIV}$) that preserve certain critical properties of the original continuum differential operators ($-K$grad and div, respectively). For the linear diffusion problem, this means that the mimetic discretizations mimic the Gauss divergence theorem to enforce the local conservation law, the symmetry between the continuous gradient and divergence operators, $\mathcal{GRAD}^* = \mathcal{DIV}$, to guarantee symmetry and positivity of the discrete operator $\mathcal{DIV}\,\mathcal{GRAD}^*$, and the null spaces of the involved operators to prove stability of the discretization [1, 2].

The current approach is based on the divide and conquer principle. Firstly the local discrete operator are constructed. Each mesh cell is considered as a separate domain and an independent discretization is generated. The normal component of the vector function $\mathbf{F}$ is computed at the face centers and the pressure unknowns p is computed at the centers of faces and mesh elements. Secondly we impose boundary conditions and interface conditions between neighbouring elements. Imposing interface condition we require a continuity for p and the normal component of $\mathbf{F}$ across the polyhedral faces. The similar approach in 2D case was described in [3].

We have proved that the resulting discretization is exact for piecewise linear solutions p (piecewise constant $\mathbf{F}$) on the unstructured polyhedral meshes with planar faces. Moreover, the solution algorithm produces a symmetric positive definite system of linear equations (semi-positive definite in the case of Neumann boundary conditions) which can be solved with efficient iterative methods.

The computational results demonstrate the flexibility and accuracy of our approximation method. For sufficiently smooth solutions, we achieve the second order convergence for scalar unknown p and at least first order for the vector unknown $\mathbf{F}$ on different types of polyhedral meshes which faces are close to planar faces. Should be noted that for smooth meshes, the polyhedron faces are close to planar faces.

In the case of polyhedral meshes with strongly curved faces, an approximation of the curved faces by planar faces is the only known way to achieve second order convergence rate for the scalar variable and the first order convergence rate for the vector variable.

References

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[3] Y. Kuznetsov, K. Lipnikov, M. Shashkov. Mimetic finite difference method on polygonal meshes for diffusion-type problems. Los Alamos National Laboratory, Report LA-UR-03-9168, to appear in *Comp. Geosciences*, 2004.